Technical note: The Optimality Conditions for Continuous Demand Distributions with Independent Increments

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We consider the optimality condition used by Gallego and van Ryzin (1994) for Poisson demands with finite time horizon as well as the optimality condition introduced by Araman and Caldentey (2009) for Poisson demands with stopping (or infinite) time horizons. We extend these two demand optimality conditions from Poisson to any arbitrary continuous demand distribution with mean λt and variance $\sigma^2 t$. We consider both finite, and stopping time horizons cases.

Key words: Stochastic Control; Poisson Demand; Continuos Demand; Stopping Time; Optimality Condition

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1. Introduction

Modeling the correct demand distribution of customers is of significant importance as the type of the demand process can considerably impact the suggested policies. A check of real demand processes shows that the customers' demand process can be frequently non-Poisson for a variety of reasons. For instance, one of the needed assumptions made in the Poisson process is that the average of the demand is equal to its variance. Nevertheless, in reality this assumption does rarely hold.

In this technical note, we consider the optimality condition applied by Gallego and van Ryzin (1994) to characterize Poisson demands with finite time horizon as well as the optimality condition introduced by Araman and Caldentey (2009) for Poisson demands with a stopping (or infinite) time horizon. We extend these two demand optimality conditions from Poisson to any arbitrary continuous distribution with mean λt and variance $\sigma^2 t$ at time t. We consider both finite and stopping time horizons for the demand. We show that in both extensions an extra second order term appears in the optimality condition that explains the adjustment needed when the demand becomes continuous. The appearance of the second order term is important as it changes the solution of the Bellman equation. Hence, the optimal operational policies may significantly alter accordingly.

2. Optimality Condition for a Finite Deterministic Time Horizon

Consider that the stochastic demand process X_t follows an arbitrary continuous distribution C with mean $\lambda_t(X_t, u_t, t)t$ and variance $\sigma_t^2(X_t, u_t, t)$, i.e.,

$$X_t \sim C(\lambda_t(X_t, u_t)t, \sigma_t(X_t, u_t)\sqrt{t}),$$

where u_t is the control variable[1]. Then, the total revenue gained through the demand from the initial time $t = 0$ to the terminal time $t = T$ is

$$J^*(x,T) = \sup_{u_t} \mathbb{E}_X [\int_0^T f_0(X_t, u_t) dX_t], \tag{1}$$

[1] For instance, at the simplest case where $\lambda_t(u_t) = u_t$ we aim to control the average of demand.

where $f_0(X_t,u_t)$ is the seller's revenue associated with the stochastic demand X_t, $0 \le t \le T$. Clearly, X_t can be re-expressed as

$$X_t = \lambda_t(X_t,u_t)t + \sigma_t(X_t,u_t)\widehat{X}_t,$$

where $\widehat{X}_t$ is a new process with the mean scaled to zero and variance t. Thus, based on Doob–Meyer's decomposition theorem (e.g., see Protter (2005)), the process dX_t can be uniquely expressed as

$$dX_t = \lambda_t(X_t,u_t)dt + \sigma_t(X_t,u_t)d\widehat{X}_t, \tag{2}$$

where the increment process $d\widehat{X}_t$ has the mean 0 and variance dt. In order to show the reason that the variance of the increment process $d\widehat{X}_t$ is equal to dt, i.e., $var[d\widehat{X}_t] = dt$, we assume that the process $\widehat{X}_t$ has independent increments, i.e., $cov(d\widehat{X}_t, d\widehat{X}_s) = 0$ for any $t \ne s$. Next, we fix an arbitrarily fine partition $\{t_0,t_1,...,t_n\}$ of the interval $[0,T]$, say into n subintervals all of equal length Δt with $\Delta t \to 0$, so that $t_i = t_0 + i\Delta t$. We shall call Δt the mesh of the subdivision. Furthermore, we note that $d\widehat{X}_t = \Delta\widehat{X}_t = \widehat{X}_{t+\Delta t} - \widehat{X}_t$ with $\Delta t \to 0$. Let us write ν the common value of the variance of the increment $\Delta\widehat{X}_t = \widehat{X}_{t+\Delta t} - \widehat{X}_t$. Thus, $\nu = var[\Delta\widehat{X}_t] = \mathbb{E}[(\Delta\widehat{X}_t)^2] - (\mathbb{E}[\Delta\widehat{X}_t])^2$ and hence since $\mathbb{E}[\Delta\widehat{X}_t] = 0$, we have for all t

$$\nu = \mathbb{E}[(\Delta\widehat{X}_t)^2].$$

We thus have by the additivity of variance over the independent increments that

$$\begin{aligned}
var[(\widehat{X}_T - \widehat{X}_0)] &= \sum_{i=1}^{n} var[(\widehat{X}_{T-i\Delta t} - \widehat{X}_{T-(i+1)\Delta t})] \\
&= \sum_{i=0}^{n} var[\Delta\widehat{X}_{t_i}] = \sum_{i=0}^{n} \mathbb{E}[(\Delta\widehat{X}_{t_i})^2] \\
&= n\nu = vT/\Delta t.
\end{aligned}$$

If in the limit as $\Delta t \to 0$, the limiting random variable $\widehat{X}_t$ is to have a finite variance at $t = T$ and $t = 0$, the limit of $vT/\Delta t$ must be also finite. This conclusion, which results from the independence of increments $\Delta\widehat{X}_{t_i}$ and the requirement of finite variance at each time, leads to the natural following standardization of the limiting process. A straightforward verification shows that if

$$\lim \frac{var(\Delta\widehat{X}_t)}{\Delta t} \to 1, \text{ as } \Delta t \to 0,$$

then the process $\widehat{X}_t$ has finite variance under C and also $var(\widehat{X}_t) = t$ for any arbitrary value of t. As a result $var(d\widehat{X}_t) = dt$.

Next, we consider that a straightforward application of (2) to the optimal revenue function (1) gives

$$J^*(x,T) = \sup_{u_t} \mathbb{E}_X \begin{pmatrix} \int_0^T f_0(X_t,u_t)\lambda_t(X_t,u_t)dt \\ + \int_0^T f_0(X_t,u_t)\sigma_t(X_t,u_t)d\widehat{X}_t) \end{pmatrix}. \tag{3}$$

Note that in the above formula, since $d\widehat{X}_t$ is a martingale with $\mathbb{E}[d\widehat{X}_t] = 0$ and $f_0(X_t,u_t)\sigma_t(X_t,u_t)$ is bounded, the term $\int_0^T f_0(X_t,u_t)\sigma_t(X_t,u_t)d\widehat{X}_t)$ is a martingale transform of the process $\widehat{X}_t$ and hence is itself a martingale. Therefore its expected value is zero. That is,

$$\mathbb{E}_X[\int_0^T f_0(X_t,u_t)\sigma_t(X_t,u_t)d\widehat{X}_t)] = 0. \tag{4}$$

Hence, the revenue function can be re-expressed as

$$J^*(x,T) = \sup_{u_t} \mathbb{E}_X[\int_0^T f_0(X_t,u_t)\lambda_t(X_t,u_t)dt], \tag{5}$$

where $x = X_0$. Our goal is now to derive the optimality conditions from equation (5). First of all, from equation (5), we establish that

$$J^*(x,T) = \sup_{u_t} \mathbb{E}_X \begin{pmatrix} \int_0^{\Delta t} f_0(X_t,u_t)\lambda_t(X_t,u_t)dt \\ + \int_{\Delta t}^T f_0(X_t,u_t)\lambda_t(X_t,u_t)dt \end{pmatrix}. \tag{6}$$

The essential observation is that, using the First Mean Value Theorem for integrals (see, e.g., Jeffreys and Jeffreys (2000)), the expected value $\mathbb{E}_X[\int_0^{\Delta t} f_0(X_t,u_t,t)\lambda_t(X_t,u_t,t)dt]$ can be expressed as

$$\mathbb{E}_X[\int_0^{\Delta t} f_0(X_t,u_t)\lambda_t(X_t,u_t)dt] = f_0(x,u)\lambda(x,u)\Delta t + o_1(\Delta t), \tag{7}$$

where $u = u_s$ is a control function defined for $0 \le s \le \Delta t$, and the last term $o_1(\Delta t)$ is a function of Δt that has the property that $o_1(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$. Now, the crucial use of the the law of repeated expectations (Tower Property) (see, e.g., Williams (1991)) gives

$$J^*(x,T) = \sup_{u_t} \begin{pmatrix} f_0(x,u)\lambda(x,u)\Delta t + \\ \mathbb{E}_X\mathbb{E}_{X,\Delta t}[\int_{\Delta t}^T f_0(X_t,u_t)\lambda_t(X_t,u_t)dt] + o_1(\Delta t) \end{pmatrix}. \tag{8}$$

In addition, an easy application of equation (7), and the change of variable $\theta_t = t - \Delta t$ to the second term $\mathbb{E}_{X,\Delta t}[\int_{\Delta t}^T f_0(X_t,u_t)\lambda_t(X_t,u_t)dt]$ gives

$$\begin{aligned} &\mathbb{E}_{X,\Delta t}[\int_0^{T-\Delta t} f_0(X_{\theta+\Delta t},u_{\theta+\Delta t})\lambda_{\theta+\Delta t}(x,u_{\theta+\Delta t})d\theta_t] \\ &= J(X_{\Delta t},T-\Delta t) = J(x-\Delta X,T-\Delta t). \end{aligned} \tag{9}$$

The key observation in obtaining the second equality is noting that $X_{\Delta t} = X_0 - \Delta X = x - \Delta X$. Thus, the optimality condition reduces to

$$J^*(x,T) = \mathbb{E}_X[f_0(x,u^*)\lambda(x,u^*)\Delta t + J^*(x-\Delta X,T-\Delta t) + o_1(\Delta t)], \tag{10}$$

where u^* belongs to the optimal control trajectory. Now, by applying the two dimensional Taylor expansion to $J^*(x-\Delta X,T-\Delta t)$ and replacing in equation (10), we find

$$J^*(x,T) = \mathbb{E}_X \begin{pmatrix} f_0(x,u^*)\lambda(x,u^*)\Delta t + o_1(\Delta t) + J^*(x,T) \\ -J_T^{*\prime}\Delta t - J_x^{*\prime}\Delta X - \frac{1}{2}J_x^{*\prime\prime}(\Delta X)^2 + o_2(\Delta t) \end{pmatrix}. \tag{11}$$

Denoting $o_1(\Delta t) + o_2(\Delta t)$ by $o(\Delta t)$, we are now ready to invoke equation (2) in order to reduce equation (11) as

$$0 = \mathbb{E}_X \begin{pmatrix} f_0(x,u^*)\Delta t - J_T^{*\prime}\Delta t - J_x^{*\prime}\lambda(x,u^*)\Delta t - J_x^{*\prime}\Delta\widehat{X} \\ -\frac{1}{2}J_x^{*\prime\prime}(\lambda(x,u^*)\Delta t + \sigma(x,u^*)\Delta\widehat{X})^2 + o(\Delta t) \end{pmatrix}. \tag{12}$$

Note that an essential observation is that $\mathbb{E}[(\Delta\widehat{X})^2] = var[\Delta\widehat{X}] = \Delta t$, which reduces equation (12) to

$$0 = f_0(x,u^*)\Delta t - J_T^{*\prime}\Delta t - J_x^{*\prime}\lambda(x,u^*)\Delta t - \frac{1}{2}J_x^{*\prime\prime}\sigma^2(x,u^*)\Delta t + o(\Delta t). \tag{13}$$

Finally, dividing both sides of equation (13) by Δt and passing to the limit as $\Delta t \to 0^+$ gives optimality condition as follows:

$$J_T^{*\prime} = f_0(x,u^*) - J_x^{*\prime}\lambda(x,u^*) - \frac{1}{2}J_x^{*\prime\prime}\sigma^2(x,u^*).$$

3. Optimality Condition for a Stopping Time

In this section, we examine an extended version of the optimality condition considered by Gallego and van Ryzin (1994), which was applied by Araman and Caldentey (2009) for characterizing Poisson demands when the time horizon is a stopping time[2]. In order to start, we consider the modified revenue function introduced by Araman and Caldentey (2009) as follows

$$J^*(x) = \sup_{u_t,\theta} \mathbb{E}_{X,\tau}[\int_0^{\tau} e^{-rt} f_0(X_t,u_t)dt + e^{-r\tau}R]. \tag{14}$$

$$\begin{aligned} &\text{Subject to:} \\ &dX_t = \lambda_t(X_t,u_t)dt + \sigma_t(X_t,u_t)d\widehat{X}_t, \\ &\tau_\theta(x) = \inf\{t \geq 0 : X_t = \theta\}. \end{aligned} \tag{15}$$

In the above formula $x = X_0$ is the realization of the initial demand's value at time $t = 0$. In addition, $\tau = \tau_\theta(X_0) = \tau_\theta(x)$ is the stopping time by reference to the underlying stochastic demand process X_t reaching a prescribed level θ, which is to be chosen optimally. We suppose that $X_0 = x \neq \theta$. r is the discount factor, and R is the salvage value received by the seller at the stopping time $\tau_\theta(x)$. With an argument similar to the one used in the previous section, we find that

$$J^*(x) = \sup_{u_t,\theta} \mathbb{E}_{X,\tau}\begin{pmatrix} f_0(x,u)\Delta t + o_1(\Delta t) \\ +\int_{\Delta t}^{\tau} e^{-rt} f_0(X_t,u_t)dt + e^{-r\tau}R \end{pmatrix}. \tag{16}$$

To obtain equation (16), we used the First Mean Value Theorem as stated in equation (7). Now, an easy change of variable $h = t - \Delta t$ gives

$$J^*(x) = \sup_{u_t,\theta}\begin{pmatrix} f_0(x,u)\Delta t + o_1(\Delta t) \\ +\mathbb{E}_{X,\tau}[\int_0^{\tau-\Delta t} e^{-(h+\Delta t)r} f_0(X_{h+\Delta t},u_{h+\Delta t})dh + e^{-r\tau}R] \end{pmatrix}. \tag{17}$$

Setting $\overline{\tau} \triangleq \tau - \Delta t$, $\overline{X}_h \triangleq X_{h+\Delta t}$, $\overline{u}_h \triangleq u_{h+\Delta t}$, and using the law of repeated expectations (Tower property), we find that

$$J^*(x) = \begin{pmatrix} f_0(x,u^*)\Delta t + o_1(\Delta t) \\ +e^{-r\Delta t}\mathbb{E}_{X,\tau}\mathbb{E}_{\overline{X},\overline{\tau}}[\int_0^{\overline{\tau}^*} e^{-hr} f_0(\overline{X}_h,\overline{u}_h)dh + e^{-r\tau^*}R] \end{pmatrix}, \tag{18}$$

where u^* is the optimal control trajectory and $\tau^* = \inf\{t \geq 0 : X_t = \theta^*\}$ is the stopping time when the stochastic demand process X_t reaches the prescribed optimal level θ^*. Furthermore, it is easy to observe that

$$J^*(X_{\Delta t}) = \mathbb{E}_{\overline{X},\overline{\tau}}[\int_0^{\overline{\tau}^*} e^{-hr} f_0(\overline{X}_h,\overline{u}_h)dh + e^{-r\tau^*}R].$$

Thus, replacing in equation (18) gives

$$J^*(x) = f_0(x,u^*)\Delta t + o_1(\Delta t) + e^{-r\Delta t}\mathbb{E}_{X,\tau}[J^*(X_{\Delta t})]. \tag{19}$$

The essential observation in equation (19) is that

$$X_{\Delta t} = X_0 - \Delta X = x - \Delta X, \tag{20}$$

$$\tau(X_{\Delta t}) = \tau(x) - \Delta t = \overline{\tau}, \tag{21}$$

$$e^{-r\Delta t} = 1 - r\Delta t + o_2(\Delta t). \tag{22}$$

[2] Note that considering an infinite time horizon follows the same lines of proof with the stopping time horizon and leads to the same optimality condition.

Applying equations (20)-(22) to equation (19), we find

$$J^*(x) = f_0(x,u^*)\Delta t + o_1(\Delta t) + (1 - r\Delta t + o_2(\Delta t))\mathbb{E}_{X,\tau}[J^*(x - \Delta X)].$$

A straightforward application of one dimensional Taylor's expansion and representing the sum of all error terms with the single term $o(\Delta t)$ gives

$$J^*(x) = \begin{pmatrix} f_0(x,u^*)\Delta t + o(\Delta t) \\ (1 - r\Delta t)\mathbb{E}_{X,\tau}[J^*(x) - J^{*\prime}(x)\Delta X + \frac{1}{2}J^{*''}(x)\,(\Delta X)^2] \end{pmatrix}.$$

Noticing that $\mathbb{E}[\Delta X] = \lambda(x,u)\Delta t$ and $var[\Delta X] = \sigma^2(x,u)\Delta t$, $J^*(x)$ reduces to

$$J^*(x) = \begin{pmatrix} J^*(x) + f_0(x,u^*)\Delta t - J^*(x)r\Delta t - \\ J^{*\prime}(x)\lambda(x,u)\Delta t + \frac{1}{2}J^{*''}(x)\sigma^2(x,u)\Delta t + o(\Delta t) \end{pmatrix}.$$

Finally, dividing both sides by Δt and passing to the limit as $\Delta t \to 0^+$ we obtain the optimality condition as follows

$$0 = f_0(x,u^*) - J^*(x)r - J^{*\prime}(x)\lambda(x,u) + \frac{1}{2}J^{*''}(x)\sigma^2(x,u).$$

4. Conclusion

In this technical note, we extended the optimality condition used by Gallego and van Ryzin (1994) to characterize Poisson demands with finite time horizon and also the optimality condition used by Araman and Caldentey (2009) for Poisson demands with a stopping (or infinite) time horizon to any arbitrary continuous distribution with mean λt and variance $\sigma^2 t$ at time t. As observed, in both extensions, an *extra second order* term appears in the optimality condition, which is a function of the demand's variance. This extra term explains the "adjustment" needed when the demand process is continuous. This adjustment changes the Bellman equation's solutions and as a result the optimal operational policies would change accordingly.

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